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Twisted conformal algebra *so*(4, 2)

N Aizawa[1], **F J Herranz**[2], **J Negro**[3] **and M A del Olmo**[3]

[1] Department of Applied Mathematics, Osaka Women's University, Sakai, 590-0035 Osaka, Japan
[2] Departamento de Física, Universidad de Burgos, E-09006 Burgos, Spain
[3] Departamento de Física Teórica, Universidad de Valladolid, E-47011 Valladolid, Spain

E-mail: aizawa@appmath.osaka-wu.ac.jp, fjherranz@ubu.es, jnegro@fta.uva.es and olmo@fta.uva.es

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Abstract

A new twisted deformation, $U_z(so(4,2))$, of the conformal algebra of the (3 + 1)-dimensional Minkowskian spacetime is presented. This construction is provided by a classical r-matrix spanned by ten Weyl–Poincaré generators, which generalizes non-standard quantum deformations previously obtained for $so(2,2)$ and $so(3,2)$. However, by introducing a conformal null-plane basis it is found that the twist can indeed be supported by an eight-dimensional carrier subalgebra. By construction the Weyl–Poincaré subalgebra remains as a Hopf subalgebra after deformation. Non-relativistic limits of $U_z(so(4,2))$ are shown to be well defined and they give rise to new twisted conformal algebras of Galilean and Carroll spacetimes. Furthermore a difference-differential massless Klein–Gordon (or wave) equation with twisted conformal symmetry is constructed through deformed momenta and position operators. The deformation parameter is interpreted as the lattice step on a uniform Minkowskian spacetime lattice discretized along two basic null-plane directions.

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1. Introduction

Quantum groups have been applied to obtain deformations of spacetime symmetries in the search for new deformed relativistic theories beyond the usual Lie symmetries, where deformation parameters have been interpreted as fundamental quantities in the field of quantum gravity at the Planck scale (see e.g. [1]). Amongst the quantum kinematical algebras we can distinguish between standard or quasitriangular type [2–9] and non-standard or triangular type [10–12]. The non-standard deformations are twisted quantum algebras [13]. For all of them, deformed coproducts indicate some broken symmetry related to some kind of interaction or correlation of the elementary systems [14].

In the last few years great activity has developed to obtain explicitly non-standard deformations of classical Lie algebras. Amongst the new recent twists that have been constructed, we can mention the so-called extended twists [15], peripheric twists [16, 17], quantum Jordanian twists [18] or parabolic twists [19]. Another possibility is to obtain a sequence of twists by composing chains of twists [20, 21]. In general, all of these twist deformations use Cartan–Weyl bases that, although well adapted for semisimple Lie algebras, in our opinion present two main objections. On the one hand, they preclude a clear application of contraction theories which are completely necessary, for instance, if non-relativist limits are searched. On the other, they do not provide, in principle, natural physical interpretations of deformation parameters. These and other reasons underlie the constructions of non-standard quantum deformations in other physical bases such as kinematical and conformal ones, that try to establish deformations of relativistic and non-relativistic spacetime symmetries. In this context we find several twisted deformations for $so(2,2)$ [10, 22], $so(3,2)$ [23, 24], and very recently for $so(4,2)$ [25, 26], as well as for other real forms together with their contractions. Moreover, in [27–29] the twist introduced in [30] is particularized to kinematical algebras. In particular, in [27, 30] it is proved that the twisted deformation of $so(3,2)$ presented there is equivalent to that introduced in [24].

The aim of this paper is to construct a new twisted deformation of the real Lie algebra $so(4,2)$ in a conformal framework that extends previously known results for lower dimensional cases as well as to study physically meaningful contractions and interpretations of the deformation parameter.

The paper is organized as follows. In section 2 we briefly review the basics on twisted deformations. Section 3 is devoted to a description of the conformal Lie algebra $so(4,2)$ in order to set up the notation used throughout the paper as well as to fix two suitable bases for our purposes: the usual conformal Minkowskian basis and a 'conformal null-plane basis' defined as a natural extension of the Poincaré basis [31]; the Weyl–Poincaré subalgebra $\mathcal{WP} \subset so(4,2)$ is identified within the latter basis.

In section 4 we generalize the classical r-matrices of the non-standard quantum deformations of $so(2,2)$ [10] and $so(3,2)$ [24] to the $so(4,2)$ case. The resulting classical r-matrix is spanned by ten Weyl–Poincaré generators (all but a rotation one), however, by writing it in the conformal null plane basis we find that the carrier subalgebra $\mathcal{L}$ is actually an eight-dimensional (8D) one. Next we introduce the associated conformal twisting element which is a factorizable extended twist so that the Hopf structure and universal $\mathcal{R}$-matrix of the twisted algebra $U_z(so(4,2))$ is then computed (the deformation parameter z is such that $\mathrm{e}^z \equiv q$). We remark that, by construction, both the carrier and Weyl–Poincaré subalgebras remain as Hopf subalgebras after deformation in such a manner that $U_z(\mathcal{L}) \subset U_z(\mathcal{WP}) \subset U_z(so(4,2))$.

A nonlinear change of basis allows us to rewrite the Hopf structure of $U_z(\mathcal{WP})$ in terms of the usual conformal basis in section 5. As a first byproduct, the twisting element and the universal $\mathcal{R}$-matrix of $U_z(so(4,2))$ are expressed in this usual basis in a very 'compact' form. Another relevant consequence is that in this last form, well-defined non-relativistic limits (contractions) to twisted conformal Galilei and Carroll algebras are applied in section 6; the latter can alternatively be seen as the $(4+1)$D version of the $(3+1)$D null-plane quantum Poincaré algebra obtained in [11, 12]. For both cases we obtain the contracted Hopf structures, twisting elements as well as universal $\mathcal{R}$-matrices in an explicit and close form.

A 'twisted analogue' of the wave or massless Klein–Gordon equation is deduced in section 7 with twisted conformal invariance by introducing some appropriate difference-differential momenta and position operators. Such a construction is performed in the usual

Minkowskian spacetime coordinates as well as in the null-plane ones. Both systems convey some kind of discretization of spacetime, although the latter exhibits more clearly a physical interpretation of the deformation parameter z as the lattice constant on a uniform relativistic spacetime lattice discretized along *two* distinguished null-plane directions. Furthermore, from a dynamical point of view within the null-plane framework we find that z may also be interpreted as a fundamental constant related to a time evolution parameter.

Some conclusions and remarks close the paper where we establish an explicit relationship between our carrier subalgebra and twisting element and those introduced by Kulish and Lyakhovsky [30] in relation to $sl(4)$ by profiting from the isomorphism as complex forms of $sl(4) \simeq so(6)$. We also comment on the new κ-conformal classical r-matrices obtained by Lukierski *et al* in [25].

2. Twisted Hopf algebras

Let $\mathcal{A}(m, \Delta, \epsilon, S)$ be a Hopf algebra with multiplication $m: \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}$, coproduct $\Delta: \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}$, counit $\epsilon: \mathcal{A} \rightarrow C$ and antipode $S: \mathcal{A} \rightarrow \mathcal{A}$. By using an invertible element (twisting) $\mathcal{F} \in \mathcal{A} \otimes \mathcal{A}$, such that $\mathcal{F} = \sum f_i^{(1)} \otimes f_i^{(2)}$, we can transform the original Hopf algebra into a new one, called the *twisted Hopf algebra* [13] $\mathcal{A}_{\mathcal{F}}(m, \Delta_{\mathcal{F}}, \epsilon, S_{\mathcal{F}})$ endowed with the same multiplication and counit, but different coproduct and antipode given by

$$\Delta_{\mathcal{F}}(a) = \mathcal{F}\Delta(a)\mathcal{F}^{-1} \qquad S_{\mathcal{F}}(a) = vS(a)v^{-1} \qquad v = \sum f_i^{(1)} S\big(f_i^{(2)}\big) \qquad \forall a \in \mathcal{A}. \tag{2.1}$$

The *twisting element* $\mathcal{F}$ has to verify the following conditions

$$(\epsilon \otimes \mathrm{id})(\mathcal{F}) = (\mathrm{id} \otimes \epsilon)(\mathcal{F}) = 1 \qquad \mathcal{F}_{12}(\Delta \otimes \mathrm{id})(\mathcal{F}) = \mathcal{F}_{23}(\mathrm{id} \otimes \Delta)(\mathcal{F}) \tag{2.2}$$

where $\mathcal{F}_{12} = \sum f_i^{(1)} \otimes f_i^{(2)} \otimes 1$, $\mathcal{F}_{23} = \sum 1 \otimes f_i^{(1)} \otimes f_i^{(2)}$, and so on. The first relation of (2.2) is a normalization condition and follows from the second one modulo a non-zero scale factor. Hence the $\mathcal{R}$-matrix associated with the twisted algebra starting either from a non-deformed Hopf algebra or from a deformed one (with universal $\mathcal{R}$-matrix $\mathcal{R}$) is given, in this order, by

$$\mathcal{R}_{\mathcal{F}} = \tau(\mathcal{F})\mathcal{F}^{-1} \qquad \mathcal{R}_{\mathcal{F}} = \tau(\mathcal{F})\mathcal{R}\mathcal{F}^{-1} \tag{2.3}$$

where τ is the permutation of the tensor product factors. So, $\tau(\mathcal{F}) = \sum f_i^{(2)} \otimes f_i^{(1)}$.

It is worth noting that if $\mathcal{A}$ is a Hopf subalgebra of a Hopf algebra $\mathcal{B}$ the twist element $\mathcal{F}$ induces a twisted deformation of $\mathcal{B}$. Let $\mathcal{A} = \mathcal{U}(\mathcal{L}) \subset \mathcal{B} = \mathcal{U}(\mathcal{G})$ be the universal enveloping algebras of Lie algebras $\mathcal{L} \subset \mathcal{G}$. When $\mathcal{U}(\mathcal{L})$ is the minimal subalgebra such that the twisting element is completely defined as $\mathcal{F} \in \mathcal{U}(\mathcal{L}) \otimes \mathcal{U}(\mathcal{L})$ it is said that $\mathcal{L}$ is the *carrier subalgebra* of $\mathcal{F}$ [32].

Most of the known twisting elements are *factorizable* with respect to the coproduct. In this paper we are interested in the particular subclass of factorizable twisting elements that verify the following factorized twist equations [33]:

$$(\Delta \otimes \mathrm{id})(\mathcal{F}) = \mathcal{F}_{13}\mathcal{F}_{23} \qquad (\mathrm{id} \otimes \Delta_{\mathcal{F}})(\mathcal{F}) = \mathcal{F}_{12}\mathcal{F}_{13} \tag{2.4}$$

which combined with the quantum Yang–Baxter equation on $\mathcal{F}$ guarantee the validity of the twist equations (2.2).

Two interesting cases of this kind of factorizable twist are Jordanian and extended Jordanian twists. A *Jordanian twist* with a 2D carrier algebra generated by H and E such that $[H, E] = E$ (Borel algebra) is characterized by the canonical twisting element [34]

$$\mathcal{F}_{\mathcal{J}} = \mathrm{e}^{H \otimes \sigma} \qquad \sigma = \ln(1 + E). \tag{2.5}$$

The best known example is related to the non-standard quantization of $sl(2)$ [35].

The problem of extending the Jordanian twist was studied in [15] by considering a carrier solvable subalgebra $\mathcal{L}$ with, at least, four generators $\{H, E, A, B\}$, and non-vanishing commutation relations given by

$$[H, E] = 2E \qquad [H, A] = \alpha A \qquad [H, B] = \beta B \qquad [A, B] = \gamma E \qquad \alpha + \beta = 2. \tag{2.6}$$

There exists an *extended Jordanian twist* for $\mathcal{U}(\mathcal{L})$ given by

$$\mathcal{F}_{\mathcal{E}} = \Phi_{\mathcal{J}} \Phi_{\mathcal{E}} = \exp(H \otimes \sigma) \exp(A \otimes B\, \mathrm{e}^{-2\sigma}). \tag{2.7}$$

New twisting elements can also be constructed by means of the composition of several twists, but note that in general the composition of twists is not a twist. However, given two twists $\mathcal{F}_1$ and $\mathcal{F}_2$ with carrier algebras $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively, the composition $\mathcal{F}_2 \circ \mathcal{F}_1$ is a twist if the primitive elements of $\mathcal{L}_2$ are in the twisted algebra $\mathcal{U}_{\mathcal{F}_1}(\mathcal{L}_1)$.

3. Conformal null-plane basis for *so*(4, 2)

The Lie algebra $so(4, 2)$ can physically be interpreted either as the kinematical algebra of the (4 + 1)D anti-de Sitter spacetime or as the conformal algebra of the (3 + 1)D Minkowskian spacetime. We shall consider the conformal interpretation and profit it in order to use a 'physical' basis instead of the Cartan–Weyl one, which is the basis commonly used in the twist constructions.

In the conformal basis we have the generators of rotations J_i, dilations D, time translations P_0, space translations P_i, boosts K_i and conformal transformations C_μ (Greek and Latin labels take the standard values: $\mu = 0, 1, 2, 3$; $i = 1, 2, 3$). The non-vanishing commutation relations of $so(4, 2)$ read

$$\begin{array}{lll}
[J_i, J_j] = \epsilon_{ijk} J_k & [J_i, K_j] = \epsilon_{ijk} K_k & [J_i, P_j] = \epsilon_{ijk} P_k \\
[J_i, C_j] = \epsilon_{ijk} C_k & [K_i, K_j] = -\epsilon_{ijk} J_k & [K_i, P_i] = P_0 \\
[K_i, P_0] = P_i & [K_i, C_0] = C_i & [K_i, C_i] = C_0 \\
[P_0, C_0] = D & [P_0, C_i] = -K_i & [P_i, C_0] = K_i \\
[P_i, C_j] = -\delta_{ij} D + \epsilon_{ijk} J_k & [P_\mu, D] = -P_\mu & [C_\mu, D] = C_\mu
\end{array} \tag{3.1}$$

where hereafter sum over repeated indices should be understood.

As is well known $so(4, 2)$ contains a (3 + 1)D Poincaré subalgebra $\mathcal{P}$ spanned by the ten generators $\{J_i, P_\mu, K_i\}$. If we enlarge $\mathcal{P}$ with the dilation generator, we find the Weyl–Poincaré subalgebra $\mathcal{WP}$, that is, the Lie algebra of the group of similitudes of the Minkowskian spacetime. Hence we have the sequence of subalgebras $\mathcal{P} \subset \mathcal{WP} \subset so(4, 2)$.

In the further construction of the twisted quantum $so(4, 2)$ algebra we shall deal with a basis closely related to the null-plane basis of the Poincaré algebra $\mathcal{P}$. So, we briefly describe the structure of $\mathcal{P}$ in relation to the null-plane evolution scheme [31]. Consider the Minkowskian spacetime with coordinates $x = (x^0, x^1, x^2, x^3)$ and metric $g = (g_{\mu\nu}) = \mathrm{diag}(+, -, -, -)$. The initial state of a quantum relativistic system can be defined on a null-plane Π_n^τ determined by $n \cdot x = \tau$, where n is a light-like vector and τ a real constant. If we take the light-like vector $n = \left(\frac{1}{2}, \frac{1}{2}, 0, 0\right)$, the coordinate system naturally adapted to Π_n^τ is given by

$$x^- = n \cdot x = \tfrac{1}{2}(x^0 - x^1) = \tau \qquad x^+ = \tfrac{1}{2}(x^0 + x^1) \qquad x_T = (x^2, x^3). \tag{3.2}$$

Therefore a point $x \in \Pi_n^\tau$ will be labelled by (x^+, x^2, x^3). The particular null-plane with $\tau = 0$, Π_n^0, is invariant under the action of the boosts generated by K_1 since the *transverse* coordinates x_T are unchanged and $e^{\chi K_1}$ maps x^+ on $e^{\chi} x^+$. The Poincaré generators can be classified according to their *goodness* γ which is directly characterized by their transformation with respect to K_1, i.e.

$$[K_1, X] = \gamma X \qquad X \in \mathcal{P}. \tag{3.3}$$

This property provides the *null-plane basis* of $\mathcal{P}$ defined by

$$\begin{array}{llll} \gamma = +1: & P_+ = \frac{1}{2}(P_0 + P_1) & E_2 = \frac{1}{2}(K_2 - J_3) & E_3 = \frac{1}{2}(K_3 + J_2) \\ \gamma = 0: & K_1 \quad J_1 \quad P_2 & P_3 & \\ \gamma = -1: & P_- = \frac{1}{2}(P_0 - P_1) & F_2 = \frac{1}{2}(K_2 + J_3) & F_3 = \frac{1}{2}(K_3 - J_2). \end{array} \tag{3.4}$$

By taking into account (3.1), the non-vanishing commutation relations of $\mathcal{P}$ in this null-plane basis turn out to be

$$\begin{array}{lll} [K_1, P_+] = P_+ & [K_1, P_-] = -P_- & [P_+, F_a] = -\frac{1}{2} P_a \\ [K_1, E_a] = E_a & [K_1, F_a] = -F_a & [P_-, E_a] = -\frac{1}{2} P_a \\ [J_1, P_a] = \epsilon_{1ab} P_b & [J_1, E_a] = \epsilon_{1ab} E_b & [J_1, F_a] = \epsilon_{1ab} F_b \\ [E_a, P_a] = P_+ & [F_a, P_a] = P_- & [E_a, F_b] = \frac{1}{2}\delta_{ab} K_1 - \frac{1}{2}\epsilon_{1ab} J_1 \end{array} \tag{3.5}$$

where from now on the two indices $a, b = 2, 3$. Each set of generators with the same goodness spans a Poincaré subgroup R_γ (note that $R_{\pm 1}$ are Abelian). The subgroup with semidirect product structure $S_+ = R_0 \odot R_{+1}$ is the stability group of Π_n^0, and the three remaining generators, that close R_{-1}, move this plane. In particular, P_- transforms Π_n^0 into Π_n^τ so that $x^- = \tau$ can be considered as an evolution parameter playing the role of a *time*. Recall that the null-plane quantum Poincaré algebra [11, 12] was constructed using a null-plane basis associated with K_3.

Now we extend the above basis of $\mathcal{P}$ to a *conformal null-plane basis* of $so(4, 2)$ by adding D, C_μ:

$$\mathcal{P} = \{K_1, J_1, P_\pm, P_a, E_a, F_a\} \quad \rightarrow \quad so(4, 2) = \{H_\pm, J_1, T_\pm, T_a, A_a, B_a, G_\pm, G_a\}$$

which in terms of the initial conformal generators of $so(4, 2)$ is defined by three sets of generators:

$$\begin{array}{ll} H_+ = \frac{1}{2}(D + K_1) & H_- = \frac{1}{2}(D - K_1) \\ T_+ = \frac{1}{2}(P_0 + P_1) & T_- = \frac{1}{2}(P_0 - P_1) \\ T_2 = \frac{1}{2}(P_2 - P_3) & T_3 = \frac{1}{2}(P_2 + P_3) \\ B_2 = \frac{1}{2}(K_2 + J_3 - K_3 + J_2) & B_3 = \frac{1}{2}(K_2 + J_3 + K_3 - J_2) \end{array} \tag{3.6}$$

$$J_1 \qquad A_2 = \frac{1}{2}(K_2 - J_3 - K_3 - J_2) \qquad A_3 = \frac{1}{2}(K_2 - J_3 + K_3 + J_2) \tag{3.7}$$

$$\begin{array}{ll} G_+ = \frac{1}{2}(C_0 + C_1) & G_- = \frac{1}{2}(C_0 - C_1) \\ G_2 = \frac{1}{2}(C_2 - C_3) & G_3 = \frac{1}{2}(C_2 + C_3). \end{array} \tag{3.8}$$

The eight generators (3.6) span the (carrier) subalgebra $\mathcal{L}$ that we shall consider in the next section as support of the twisting element of $so(4, 2)$. These generators together with (3.7) give rise to the Weyl–Poincaré subalgebra $\mathcal{WP}$. Finally, the new conformal transformations (3.8) complete the conformal null-plane basis of $so(4, 2)$. Note also the following relations with the null-plane generators of $\mathcal{P}$ (3.4):

$$T_\pm = P_\pm \qquad A_2 = E_2 - E_3 \qquad A_3 = E_2 + E_3 \qquad B_2 = F_2 - F_3 \qquad B_3 = F_2 + F_3. \tag{3.9}$$

The goodness (3.3) of these new generators reads (cf (3.4)):

$$\begin{array}{llllllll} \gamma = +1: & T_+ & A_2 & A_3 & G_+ \\ \gamma = 0: & H_+ & H_- & J_1 & T_2 & T_3 & G_2 & G_3 \\ \gamma = -1: & T_- & B_2 & B_3 & G_-. \end{array} \tag{3.10}$$

Starting from (3.1), we compute the commutation relations of $so(4,2)$ in terms of the conformal null-plane basis (3.6)–(3.8). The non-vanishing commutators are displayed in three sets as follows:

- Carrier subalgebra $\mathcal{L} = \{H_\pm, T_\pm, T_a, B_a\}$:

$$\begin{array}{llll} [H_+, T_+] = T_+ & [H_+, T_a] = \frac{1}{2}T_a & [H_+, B_a] = -\frac{1}{2}B_a & [B_a, T_+] = T_a \\ [H_-, T_-] = T_- & [H_-, T_a] = \frac{1}{2}T_a & [H_-, B_a] = \frac{1}{2}B_a & [B_a, T_a] = T_-. \end{array} \tag{3.11}$$

- Commutation relations between $\mathcal{L}$ and $\{A_a, J_1\}$ (with the above ones (3.11) close $\mathcal{WP}$):

$$\begin{array}{lll} [H_+, A_a] = \frac{1}{2}A_a & [A_a, T_a] = T_+ & [J_1, A_a] = \epsilon_{1ab}A_b \\ [H_-, A_a] = -\frac{1}{2}A_a & [A_a, T_-] = T_a & [J_1, B_a] = \epsilon_{1ab}B_b \\ [J_1, T_a] = \epsilon_{1ab}T_b & [A_a, B_b] = \delta_{ab}(H_+ - H_-) - \epsilon_{1ab}J_1. & \end{array} \tag{3.12}$$

- Commutators involving the new conformal transformations $\{G_\pm, G_a\}$:

$$\begin{array}{lll} [H_+, G_-] = -G_- & [H_+, G_a] = -\frac{1}{2}G_a & [G_+, T_-] = -H_- \\ [H_-, G_+] = -G_+ & [H_-, G_a] = -\frac{1}{2}G_a & [G_-, T_+] = -H_+ \\ [G_+, T_a] = -\frac{1}{2}A_a & [G_+, B_a] = -G_a & [G_a, T_+] = \frac{1}{2}A_a \\ [G_-, T_a] = -\frac{1}{2}B_a & [G_-, A_a] = -G_a & [G_a, T_-] = \frac{1}{2}B_a \\ [G_a, A_a] = -G_+ & [G_a, B_a] = -G_- & [J_1, G_a] = \epsilon_{1ab}G_b \\ [G_a, T_b] = \frac{1}{2}\delta_{ab}(H_+ + H_-) + \frac{1}{2}\epsilon_{1ab}J_1. & & \end{array} \tag{3.13}$$

Similar to the null-plane Poincaré basis, generators with the same goodness close a $SO(4,2)$ subgroup M_γ; $M_{\pm 1}$ are again Abelian.

4. Twisting deformation of *so*(4, 2) in a conformal null-plane basis

Non-standard quantum deformations for lower dimensional conformal algebras of the Minkowskian spacetime have already been introduced by starting from the Jordanian classical r-matrix of the Borel algebra [34]. The results cover the (1 + 1)D case with $so(2,2)$ [10], the (2 + 1)D case with $so(3,2)$ [24] and also the corresponding quantum conformal algebras for the Galilean spacetime obtained by means of contraction (or the non-relativistic limit). The non-standard (triangular) classical r-matrices underlying these quantum algebras are given by

$$\begin{array}{ll} U_z(\mathcal{WP}(1+1)) \subset U_z(so(2,2)): & r = z(D \wedge P_0 + K_1 \wedge P_1) \\ U_z(\mathcal{WP}(2+1)) \subset U_z(so(3,2)): & r = z(D \wedge P_0 + K_1 \wedge P_1 + K_2 \wedge P_2) + zJ \wedge P_2 \end{array} \tag{4.1}$$

where z is the deformation parameter. By construction, the Weyl–Poincaré subalgebra remains as a Hopf subalgebra after deformation since the generators involved in the r-matrices span $\mathcal{WP}$. In fact, the presence of the dilation generator D in these r-matrices is essential in order to fulfil the classical Yang–Baxter equation. This is worth comparison with the standard (quasitriangular) quantum deformations of the Poincaré algebra, such as the κ-Poincaré [2–4], that do not require the presence of D. These kind of properties have been studied for other Drinfeld–Jimbo deformations in [36]. Recall also that other different non-standard classical

r-matrices for $so(3, 2)$ and $so(4, 2)$ in a conformal framework have been obtained in [23], and very recently a detailed study of classical r-matrices giving rise to κ-deformations of $so(4, 2)$ with κ-Weyl–Poincaré Hopf subalgebras has been presented in [25].

The generalization of the classical r-matrices (4.1) to the (3 + 1)D case reads

$$r = z(D \wedge P_0 + K_1 \wedge P_1 + K_2 \wedge P_2 + K_3 \wedge P_3) + z(J_3 \wedge P_2 - J_2 \wedge P_3) \tag{4.2}$$

which is again formed by generators of $\mathcal{WP}$. In order to deduce the corresponding twisted Hopf algebra of $so(4, 2)$, we shall adopt the conformal null-plane basis (3.6)–(3.8) associated with K_1, for which the r-matrix becomes

$$r = 2z(H_+ \wedge T_+ + H_- \wedge T_- + B_2 \wedge T_2 + B_3 \wedge T_3). \tag{4.3}$$

Hence we have a 8D carrier subalgebra $\mathcal{L}$ with generators (3.6) and non-vanishing Lie brackets (3.11) instead of the 11D subalgebra $\mathcal{WP}$.

It is worth remarking that two other algebraically equivalent classical r-matrices can be obtained starting from (4.2): keep the first term and change the second by another with permuted indices as $J_1 \wedge P_3 - J_3 \wedge P_1$ or $J_2 \wedge P_1 - J_1 \wedge P_2$. These possibilities would be related to the null-plane basis associated with K_2 or K_3, respectively.

We proceed now to obtain the coproduct for the Hopf algebra $U_z(so(4, 2))$ in three steps: first for the generators of the carrier Hopf subalgebra $U_z(\mathcal{L})$ by means of the twisting element; second for $\{A_a, J_1\}$ that complete the Hopf subalgebra $U_z(\mathcal{WP})$; and finally for the remaining conformal transformations. Once the Hopf structure of $U_z(so(4, 2))$ is found, we shall change back to the initial conformal basis, where contractions and a study of difference-differential equations can meaningfully be performed.

4.1. Twisting element and carrier Hopf subalgebra

The twisting element associated with the carrier subalgebra $\mathcal{L} = \{H_\pm, T_\pm, T_a, B_a\}$, with underlying classical r-matrix (4.3), is composed by three factors

$$\mathcal{F} = \Phi_{\mathcal{A}} \Phi_{\mathcal{E}} \Phi_{\mathcal{J}} \tag{4.4}$$

where

$$\begin{aligned} \Phi_{\mathcal{J}} &= \mathrm{e}^{H_- \otimes \sigma_-} \\ \Phi_{\mathcal{E}} &= \exp\left(-2z B_3 \otimes T_3\, \mathrm{e}^{-\frac{1}{2}\sigma_-}\right) \exp\left(-2z B_2 \otimes T_2\, \mathrm{e}^{-\frac{1}{2}\sigma_-}\right) \\ \Phi_{\mathcal{A}} &= \mathrm{e}^{H_+ \otimes \sigma_+} \end{aligned} \tag{4.5}$$

with

$$\sigma_- = \ln(1 - 2zT_-) \qquad \sigma_+ = \ln(1 - 2zT) \qquad T = T_+ + z\left(T_2^2 + T_3^2\right) \mathrm{e}^{-\sigma_-}. \tag{4.6}$$

The term $\Phi_{\mathcal{J}}$ is a Jordanian twist (2.5), $\mathcal{F}_{\mathcal{E}} \equiv \Phi_{\mathcal{E}} \Phi_{\mathcal{J}}$ is an extended twist (2.7) and $\Phi_{\mathcal{A}}$ is an additional Jordanian twist on the deformed carrier space of the extended twist. The explicit structure of $\mathcal{F}$ is the generalization of the (2 + 1)D case associated with $so(3, 2)$ introduced in [27, 28, 30].

After progressive application of the three twist factors on the primitive (non-deformed) coproducts $\Delta(X) = X \otimes 1 + 1 \otimes X$, we obtain deformed coproducts $\Delta_{\mathcal{F}}$ for the generators of $\mathcal{L}$. Useful relations for computations are given by

$$\begin{aligned} &[H_+, \sigma_-] = 0 \qquad [H_+, T] = T \qquad [H_+, \sigma_+] = 1 - \mathrm{e}^{-\sigma_+} \\ &[H_-, \sigma_-] = 1 - \mathrm{e}^{-\sigma_-} \qquad [H_-, T] = z\left(T_2^2 + T_3^2\right) \mathrm{e}^{-2\sigma_-} \\ &[H_-, \sigma_+] = -2z^2\left(T_2^2 + T_3^2\right) \mathrm{e}^{-(\sigma_+ + 2\sigma_-)} \\ &[B_a, \sigma_-] = 0 \qquad [B_a, T] = T_a\, \mathrm{e}^{-\sigma_-} \qquad [B_a, \sigma_+] = -2zT_a\, \mathrm{e}^{-(\sigma_+ + \sigma_-)}. \end{aligned}$$

Note also that the Abelian commutation relations for the translation generators ensure that all the elements $\{T_\pm, T_a, T, \sigma_+, \sigma_-\}$ commute amongst themselves. The results are as follows:

- *Jordanian twist* $\Delta_{\mathcal{J}} \equiv \Phi_{\mathcal{J}} \Delta \Phi_{\mathcal{J}}^{-1}$:

$$\begin{aligned} \Delta_{\mathcal{J}}(H_+) &= H_+ \otimes 1 + 1 \otimes H_+ & \Delta_{\mathcal{J}}(T_+) &= T_+ \otimes 1 + 1 \otimes T_+ \\ \Delta_{\mathcal{J}}(H_-) &= H_- \otimes \mathrm{e}^{-\sigma_-} + 1 \otimes H_- & \Delta_{\mathcal{J}}(T_-) &= T_- \otimes \mathrm{e}^{\sigma_-} + 1 \otimes T_- \\ \Delta_{\mathcal{J}}(B_a) &= B_a \otimes \mathrm{e}^{\frac{1}{2}\sigma_-} + 1 \otimes B_a & \Delta_{\mathcal{J}}(T_a) &= T_a \otimes \mathrm{e}^{\frac{1}{2}\sigma_-} + 1 \otimes T_a. \end{aligned} \tag{4.7}$$

- *Extended twist* $\Delta_{\mathcal{E}} = \Phi_{\mathcal{E}} \Delta_{\mathcal{J}} \Phi_{\mathcal{E}}^{-1}$:

$$\begin{aligned} \Delta_{\mathcal{E}}(H_+) &= H_+ \otimes 1 + 1 \otimes H_+ \\ \Delta_{\mathcal{E}}(H_-) &= H_- \otimes \mathrm{e}^{-\sigma_-} + 1 \otimes H_- + 2zB_2 \otimes T_2\, \mathrm{e}^{-\frac{3}{2}\sigma_-} + 2zB_3 \otimes T_3\, \mathrm{e}^{-\frac{3}{2}\sigma_-} \\ \Delta_{\mathcal{E}}(B_a) &= B_a \otimes \mathrm{e}^{-\frac{1}{2}\sigma_-} + 1 \otimes B_a \\ \Delta_{\mathcal{E}}(T_+) &= T_+ \otimes 1 + 1 \otimes T_+ - 2zT_2 \otimes T_2\, \mathrm{e}^{-\frac{1}{2}\sigma_-} - 2zT_3 \otimes T_3\, \mathrm{e}^{-\frac{1}{2}\sigma_-} \\ &\quad + 2z^2 T_- \otimes \left(T_2^2 + T_3^2\right) \mathrm{e}^{-\sigma_-} \\ \Delta_{\mathcal{E}}(T_-) &= T_- \otimes \mathrm{e}^{\sigma_-} + 1 \otimes T_- \\ \Delta_{\mathcal{E}}(T_a) &= T_a \otimes \mathrm{e}^{\frac{1}{2}\sigma_-} + \mathrm{e}^{\sigma_-} \otimes T_a. \end{aligned} \tag{4.8}$$

- *Additional Jordanian twist* $\Delta_{\mathcal{F}} = \Phi_{\mathcal{A}} \Delta_{\mathcal{E}} \Phi_{\mathcal{A}}^{-1}$:

$$\begin{aligned} \Delta_{\mathcal{F}}(H_+) &= H_+ \otimes \mathrm{e}^{-\sigma_+} + 1 \otimes H_+ \\ \Delta_{\mathcal{F}}(H_-) &= H_- \otimes \mathrm{e}^{-\sigma_-} + 1 \otimes H_- + 2zB_2 \otimes T_2\, \mathrm{e}^{-\frac{1}{2}(\sigma_+ + 3\sigma_-)} + 2zB_3 \otimes T_3\, \mathrm{e}^{-\frac{1}{2}(\sigma_+ + 3\sigma_-)} \\ &\quad + 2z^2 H_+ \otimes \left(T_2^2 + T_3^2\right) \mathrm{e}^{-(\sigma_+ + 2\sigma_-)} \\ \Delta_{\mathcal{F}}(B_a) &= B_a \otimes \mathrm{e}^{-\frac{1}{2}(\sigma_+ + \sigma_-)} + 1 \otimes B_a + 2zH_+ \otimes T_a\, \mathrm{e}^{-(\sigma_+ + \sigma_-)} \\ \Delta_{\mathcal{F}}(T_+) &= T_+ \otimes \mathrm{e}^{\sigma_+} + 1 \otimes T_+ - 2zT_2 \otimes T_2\, \mathrm{e}^{\frac{1}{2}(\sigma_+ - \sigma_-)} - 2zT_3 \otimes T_3\, \mathrm{e}^{\frac{1}{2}(\sigma_+ - \sigma_-)} \\ &\quad + 2z^2 T_- \otimes \left(T_2^2 + T_3^2\right) \mathrm{e}^{-\sigma_-} \\ \Delta_{\mathcal{F}}(T_-) &= T_- \otimes \mathrm{e}^{\sigma_-} + 1 \otimes T_- \\ \Delta_{\mathcal{F}}(T_a) &= T_a \otimes \mathrm{e}^{\frac{1}{2}(\sigma_+ + \sigma_-)} + \mathrm{e}^{\sigma_-} \otimes T_a. \end{aligned} \tag{4.9}$$

The three coproducts for T_-, which are the same, show that σ_- always remains primitive

$$\Delta_{\mathcal{J}}(\sigma_-) = \Delta_{\mathcal{E}}(\sigma_-) = \Delta_{\mathcal{F}}(\sigma_-) = \sigma_- \otimes 1 + 1 \otimes \sigma_-.$$

The element σ_+ is also primitive: $\Delta_{\mathcal{F}}(\sigma_+) = \sigma_+ \otimes 1 + 1 \otimes \sigma_+$, provided that

$$\Delta_{\mathcal{F}}(T) = T \otimes \mathrm{e}^{\sigma_+} + 1 \otimes T. \tag{4.10}$$

Therefore, the final twisted coproducts (4.9) together with commutation relations (3.11) define the twisted carrier algebra $U_z(\mathcal{L})$. Furthermore these results allow us to state:

Proposition 4.1. *The element $\mathcal{F}$ given by (4.4) is a factorizable twist that fulfils equations (2.4).*

In [15] it was already proved that the product $\Phi_{\mathcal{E}}\Phi_{\mathcal{J}}$ satisfies such equations. Hence, it is only necessary to analyse the third factor $\Phi_{\mathcal{A}}$. It can be checked that it satisfies

$$(\Delta_{\mathcal{E}} \otimes \mathrm{id})(\Phi_{\mathcal{A}}) = (\Phi_{\mathcal{A}})_{13}(\Phi_{\mathcal{A}})_{23} \qquad (\mathrm{id} \otimes \Delta_{\mathcal{F}})(\Phi_{\mathcal{A}}) = (\Phi_{\mathcal{A}})_{12}(\Phi_{\mathcal{A}})_{13} \tag{4.11}$$

which guarantees the twisting condition

$$(\Phi_{\mathcal{A}})_{12}(\Delta_{\mathcal{E}} \otimes \mathrm{id})(\Phi_{\mathcal{A}}) = (\Phi_{\mathcal{A}})_{23}(\mathrm{id} \otimes \Delta_{\mathcal{E}})(\Phi_{\mathcal{A}}). \tag{4.12}$$

In other words, $\Phi_{\mathcal{A}}$ is a Jordanian twisting element for the deformed algebra obtained by an extended twist based on the carrier subalgebra spanned by $\{H_-, T_-, T_a, B_a\}$ since $[H_+, T] = T$.

4.2. Weyl–Poincaré Hopf subalgebra

Let us compute the coproducts for J_1 and A_a. From (3.12) we first find some useful relations:

$$[J_1,\sigma_-]=[J_1,T]=[J_1,\sigma_+]=0 \qquad [A_a,\sigma_-]=-2zT_a\,\mathrm{e}^{-\sigma_-}$$
$$[A_a,T]=T_a(1-\mathrm{e}^{\sigma_+})\,\mathrm{e}^{-\sigma_-} \qquad [A_a,\sigma_+]=2zT_a(1-\mathrm{e}^{-\sigma_+})\,\mathrm{e}^{-\sigma_-}.$$

Hence J_1 remains primitive for the three twists:

$$\Delta_{\mathcal{J}}(J_1)=\Delta_{\mathcal{E}}(J_1)=\Delta_{\mathcal{F}}(J_1)=J_1\otimes 1+1\otimes J_1. \tag{4.13}$$

The three steps for the coproduct of A_a are given by

$$\begin{aligned}
\Delta_{\mathcal{J}}(A_a)&=A_a\otimes \mathrm{e}^{-\frac{1}{2}\sigma_-}+1\otimes A_a+2zH_-\otimes T_a\,\mathrm{e}^{-\sigma_-}\\
\Delta_{\mathcal{E}}(A_a)&=A_a\otimes \mathrm{e}^{-\frac{1}{2}\sigma_-}+1\otimes A_a+2zH_+\otimes T_a\,\mathrm{e}^{-\sigma_-}\\
&\quad+2zB_a\otimes T\,\mathrm{e}^{-\frac{1}{2}\sigma_-}-2z\epsilon_{1ab}J_1\otimes T_b\,\mathrm{e}^{-\sigma_-}\\
\Delta_{\mathcal{F}}(A_a)&=A_a\otimes \mathrm{e}^{\frac{1}{2}(\sigma_+-\sigma_-)}+1\otimes A_a+2zH_+\otimes T_a\,\mathrm{e}^{-(\sigma_++\sigma_-)}\\
&\quad+2zB_a\otimes T\,\mathrm{e}^{-\frac{1}{2}(\sigma_++\sigma_-)}-2z\epsilon_{1ab}J_1\otimes T_b\,\mathrm{e}^{-\sigma_-}.
\end{aligned} \tag{4.14}$$

Consequently, the previous quantum carrier algebra $U_z(\mathcal{L})$, determined by coproducts (4.9) and commutators (3.11), enlarged with coproducts (4.13) and (4.14) together with the commutation rules (3.12), determine, by construction, a new twisted Weyl–Poincaré algebra in a null-plane basis, $U_z(\mathcal{WP})$, such that $U_z(\mathcal{L})\subset U_z(\mathcal{WP})$.

4.3. Conformal transformations

In the same way, cumbersome computations would lead to the coproducts of the conformal transformations completing $U_z(so(4,2))$. We restrict ourselves to displaying the auxiliary commutation relations between the generators $\{G_\pm, G_a\}$ and the elements $\{\sigma_\pm, T\}$:

$$\begin{aligned}
&[G_+,\sigma_-]=2z\,\mathrm{e}^{-\sigma_-}H_-+2z^2\,\mathrm{e}^{-2\sigma_-}T_- \qquad [G_-,\sigma_-]=0 \qquad [G_a,\sigma_-]=-z\,\mathrm{e}^{-\sigma_-}B_a\\
&[G_+,T]=-z\,\mathrm{e}^{-\sigma_-}\{T_2A_2+T_3A_3+T_++2(T-T_+)(H_-+\mathrm{e}^{-\sigma_-})\}\\
&[G_-,T]=-H_+-z\,\mathrm{e}^{-\sigma_-}(T_2B_2+T_3B_3+T_-)\\
&[G_a,T]=\tfrac{1}{2}A_a+z\,\mathrm{e}^{-\sigma_-}\{T_a(H_++H_-)+\epsilon_{1ab}T_bJ_1+T_a\,(\mathrm{e}^{-\sigma_-}-1)+(T-T_+)B_a\}\\
&[G_+,\sigma_+]=2z^2\,\mathrm{e}^{-(\sigma_++\sigma_-)}\{T_2A_2+T_3A_3+T_++2(T-T_+)(H_-+\mathrm{e}^{-\sigma_-})\}\\
&\qquad+4z^3\,\mathrm{e}^{-(2\sigma_++\sigma_-)}\{\mathrm{e}^{-\sigma_-}(T-T_+)^2+T(T-T_+)\}\\
&[G_-,\sigma_+]=2z\,\mathrm{e}^{-\sigma_+}\{H_++z\,\mathrm{e}^{-\sigma_-}(T_2B_2+T_3B_3+T_-)\}+2z^2\,\mathrm{e}^{-2\sigma_+}\{T+\mathrm{e}^{-\sigma_-}(T-T_+)\}\\
&[G_a,\sigma_+]=-z\,\mathrm{e}^{-\sigma_+}A_a-4z^3\,\mathrm{e}^{-(2\sigma_++\sigma_-)}T_a\{T+\mathrm{e}^{-\sigma_-}(T-T_+)\}\\
&\qquad-2z^2\,\mathrm{e}^{-(\sigma_++\sigma_-)}\{T_a(H_++H_-)+\epsilon_{1ab}T_bJ_1+T_a(\mathrm{e}^{-\sigma_-}-1)+(T-T_+)B_a\}.
\end{aligned}$$

Finally, we remark that the universal quantum $\mathcal{R}$-matrix for a twisted algebra is given in terms of the twisting element by (2.3). Hence, in our particular case it reads

$$\mathcal{R}=\tau(\Phi_{\mathcal{A}})\tau(\Phi_{\mathcal{E}})\tau(\Phi_{\mathcal{J}})\Phi_{\mathcal{J}}^{-1}\Phi_{\mathcal{E}}^{-1}\Phi_{\mathcal{A}}^{-1}. \tag{4.15}$$

The first order term of the power series of $\mathcal{R}$ in z reproduces the classical r-matrix (4.3). By construction the three Hopf algebras in the sequence $U_z(\mathcal{L})\subset U_z(\mathcal{WP})\subset U_z(so(4,2))$ share the same universal $\mathcal{R}$-matrix and classical r-matrix.

5. Weyl–Poincaré Hopf subalgebra in the standard conformal basis

The twisted Weyl–Poincaré algebra $U_z(\mathcal{WP})$, previously obtained in a null-plane basis, can be written in the initial conformal basis (3.1) through the following nonlinear map

$$\begin{aligned} &P_0 = -\frac{1}{2z}(\sigma_+ + \sigma_-) && P_1 = -\frac{1}{2z}(\sigma_+ - \sigma_-) \\ &P_2 = (T_3 + T_2)\,\mathrm{e}^{-\sigma_-} && P_3 = (T_3 - T_2)\,\mathrm{e}^{-\sigma_-} \\ &D = H_+ + H_- && K_1 = H_+ - H_- \\ &K_2 = \tfrac{1}{2}(A_3 + A_2 + B_3 + B_2) && K_3 = \tfrac{1}{2}(A_3 - A_2 + B_3 - B_2) \\ &J_2 = \tfrac{1}{2}(A_3 - A_2 - B_3 + B_2) && J_3 = -\tfrac{1}{2}(A_3 + A_2 - B_3 - B_2) \end{aligned} \tag{5.1}$$

with J_1 unchanged. The limit $z \to 0$ gives rise to the change of basis (3.6) and (3.7). This map is the natural generalization of that introduced in [27, 28] for $U_z(\mathcal{WP}(2+1))$ in relation to the non-standard quantum $so(3,2)$.

By taking into account the inverse map for the commuting elements $\{T_\pm, T_a, T, \sigma_\pm\}$

$$\begin{aligned} &\sigma_- = -z(P_0 - P_1) \qquad T_- = \frac{1 - \mathrm{e}^{-zP_0}\,\mathrm{e}^{zP_1}}{2z} \qquad T_2 = \frac{1}{2}\,\mathrm{e}^{\sigma_-}(P_2 - P_3) \\ &\sigma_+ = -z(P_0 + P_1) \qquad T = \frac{1 - \mathrm{e}^{-zP_0}\,\mathrm{e}^{-zP_1}}{2z} \qquad T_3 = \frac{1}{2}\,\mathrm{e}^{\sigma_-}(P_2 + P_3) \\ &T_+ = \frac{1 - \mathrm{e}^{-zP_0}\,\mathrm{e}^{-zP_1}}{2z} - \frac{z}{2}\,\mathrm{e}^{-zP_0}\,\mathrm{e}^{zP_1}\left(P_2^2 + P_3^2\right) \end{aligned} \tag{5.2}$$

the results of the previous section and the auxiliary commutators given by

$$\begin{aligned} &[J_a, \sigma_-] = -z\epsilon_{1ab}P_b && [J_a, T] = -\tfrac{1}{2}\epsilon_{1ab}\,\mathrm{e}^{\sigma_+}P_b && [J_a, \sigma_+] = z\epsilon_{1ab}P_b \\ &[K_a, \sigma_-] = -zP_a && [K_a, T] = \left(1 - \tfrac{1}{2}\,\mathrm{e}^{\sigma_+}\right)P_a && [K_a, \sigma_+] = z(1 - 2\,\mathrm{e}^{-\sigma_+})P_a \end{aligned}$$

it can be found that the coproduct and non-vanishing commutation rules for $U_z(\mathcal{WP})$ in the usual conformal basis turn out to be

$$\begin{aligned} \Delta(P_0) &= P_0 \otimes 1 + 1 \otimes P_0 \qquad \Delta(P_1) = P_1 \otimes 1 + 1 \otimes P_1 \\ \Delta(P_a) &= P_a \otimes \mathrm{e}^{-zP_1} + 1 \otimes P_a \qquad \Delta(J_1) = J_1 \otimes 1 + 1 \otimes J_1 \\ \Delta(J_a) &= J_a \otimes \mathrm{e}^{-zP_1} + 1 \otimes J_a + zJ_1 \otimes P_a \\ \Delta(D) &= D \otimes \mathrm{e}^{zP_0}\cosh zP_1 + 1 \otimes D + K_1 \otimes \mathrm{e}^{zP_0}\sinh zP_1 + z(K_2 + J_3) \otimes \mathrm{e}^{zP_0}P_2 \\ &\quad + z(K_3 - J_2) \otimes \mathrm{e}^{zP_0}P_3 + \frac{z^2}{2}(D + K_1) \otimes \mathrm{e}^{zP_0}\,\mathrm{e}^{zP_1}\left(P_2^2 + P_3^2\right) \\ \Delta(K_1) &= K_1 \otimes \mathrm{e}^{zP_0}\cosh zP_1 + 1 \otimes K_1 + D \otimes \mathrm{e}^{zP_0}\sinh zP_1 - z(K_2 + J_3) \otimes \mathrm{e}^{zP_0}P_2 \\ &\quad - z(K_3 - J_2) \otimes \mathrm{e}^{zP_0}P_3 - \frac{z^2}{2}(D + K_1) \otimes \mathrm{e}^{zP_0}\,\mathrm{e}^{zP_1}\left(P_2^2 + P_3^2\right) \\ \Delta(K_a) &= K_a \otimes \mathrm{e}^{zP_0} + 1 \otimes K_a + \epsilon_{1ab}J_b \otimes (\mathrm{e}^{zP_0} - \mathrm{e}^{-zP_1}) - z\epsilon_{1ab}J_1 \otimes P_b \\ &\quad + z(D + K_1) \otimes \mathrm{e}^{zP_0}\,\mathrm{e}^{zP_1}P_a \end{aligned} \tag{5.3}$$

$$
\begin{aligned}
&[J_i, J_j] = \epsilon_{ijk} J_k \qquad [J_i, K_j] = \epsilon_{ijk} K_k \qquad [K_i, K_j] = -\epsilon_{ijk} J_k \qquad [J_1, P_j] = \epsilon_{1jk} P_k \\
&[J_a, P_1] = \epsilon_{a1b} P_b \qquad [J_a, P_b] = \epsilon_{1ab} \left(\frac{1 - \mathrm{e}^{-2zP_1}}{2z} + \frac{z}{2} \left(P_b^2 - P_a^2 \right) \right) + z\delta_{ab}(\epsilon_{1bc} P_b P_c) \\
&[K_1, P_0] = \frac{1}{z} \mathrm{e}^{zP_0} \sinh z P_1 - \frac{z}{2} \mathrm{e}^{zP_0} \mathrm{e}^{zP_1} \left(P_2^2 + P_3^2 \right) \\
&[K_1, P_1] = \frac{1}{z} \left(\mathrm{e}^{zP_0} \cosh z P_1 - 1 \right) - \frac{z}{2} \mathrm{e}^{zP_0} \mathrm{e}^{zP_1} \left(P_2^2 + P_3^2 \right) \\
&[K_a, P_0] = \mathrm{e}^{zP_0} \mathrm{e}^{zP_1} P_a \qquad [K_a, P_1] = (\mathrm{e}^{zP_0} \mathrm{e}^{zP_1} - 1) P_a \qquad\qquad (5.4) \\
&\qquad [K_1, P_a] = (1 - \mathrm{e}^{zP_0} \mathrm{e}^{-zP_1}) P_a \\
&[K_a, P_b] = \delta_{ab} \left\{ \frac{1}{z} \mathrm{e}^{-zP_1} (\mathrm{e}^{zP_0} - \cosh z P_1) + \frac{z}{2} \epsilon_{1bc}^2 \left(P_b^2 - P_c^2 \right) \right\} + z\epsilon_{1ab}^2 P_a P_b \\
&[D, P_0] = \frac{1}{z} (\mathrm{e}^{zP_0} \cosh z P_1 - 1) + \frac{z}{2} \mathrm{e}^{zP_0} \mathrm{e}^{zP_1} \left(P_2^2 + P_3^2 \right) \\
&[D, P_1] = \frac{1}{z} \mathrm{e}^{zP_0} \sinh z P_1 + \frac{z}{2} \mathrm{e}^{zP_0} \mathrm{e}^{zP_1} \left(P_2^2 + P_3^2 \right) \qquad [D, P_a] = \mathrm{e}^{zP_0} \mathrm{e}^{-zP_1} P_a,
\end{aligned}
$$

(recall that $a, b, c = 2, 3$ while $i, j, k = 1, 2, 3$). In this way we find the (3 + 1)D version of the non-standard quantum $\mathcal{WP}$ algebra obtained in [24].

Furthermore, the map (5.1) allows us to write the universal $\mathcal{R}$-matrix of both $U_z(\mathcal{WP})$ and $U_z(so(4, 2))$ (4.15) in this standard conformal basis provided that the three factors (4.5) of $\mathcal{F}$ read

$$
\begin{aligned}
&\Phi_{\mathcal{J}} = \exp\left(-\frac{z}{2} D \otimes P_0\right) \exp\left(\frac{z}{2} D \otimes P_1\right) \exp\left(\frac{z}{2} K_1 \otimes P_0\right) \exp\left(-\frac{z}{2} K_1 \otimes P_1\right) \\
&\Phi_{\mathcal{E}} = \exp\left\{-z(K_2 + J_3) \otimes \mathrm{e}^{-\frac{z}{2}P_0} \mathrm{e}^{\frac{z}{2}P_1} P_2\right\} \exp\left\{-z(K_3 - J_2) \otimes \mathrm{e}^{-\frac{z}{2}P_0} \mathrm{e}^{\frac{z}{2}P_1} P_3\right\} \qquad (5.5) \\
&\Phi_{\mathcal{A}} = \exp\left(-\frac{z}{2} D \otimes P_0\right) \exp\left(-\frac{z}{2} D \otimes P_1\right) \exp\left(-\frac{z}{2} K_1 \otimes P_0\right) \exp\left(-\frac{z}{2} K_1 \otimes P_1\right).
\end{aligned}
$$

In the computations we have used the following expression valid whenever $[A, B] = 0$:

$$
\exp\{(A + B) \otimes C\} = \exp\{A \otimes C\} \exp\{B \otimes C\}.
$$

Moreover, the twisting element $\mathcal{F}$ (4.4) of $U_z(so(4, 2))$ can be obtained in a compact form by applying the expression

$$
\exp(\alpha A \otimes B) \exp(\beta C \otimes D) = \exp\{\beta C \otimes D\, \mathrm{e}^{\alpha\gamma B}\} \exp(\alpha A \otimes B) \qquad \alpha, \beta, \gamma \in \mathbb{C}
$$

valid whenever $[A, C] = \gamma C$ and $[B, D] = 0$. The resulting $\mathcal{F}$ is given by

$$
\begin{aligned}
\mathcal{F} = {} & \exp\{-z(K_3 - J_2) \otimes \mathrm{e}^{zP_1} P_3\} \exp\{-z(K_2 + J_3) \otimes \mathrm{e}^{zP_1} P_2\} \\
& \times \exp(-zK_1 \otimes P_1) \exp(-zD \otimes P_0) \qquad (5.6)
\end{aligned}
$$

which gives rise to the $\mathcal{R}$-matrix $\mathcal{R} = \tau(\mathcal{F})\mathcal{F}^{-1}$ with underlying classical r-matrix given by (4.2).

6. Quantum contractions

The twisted Minkowskian conformal algebra has well-defined non-relativistic limits to quantum conformal algebras of either Galilean or Carroll spacetimes [37], denoted $\mathcal{G}$ and $\mathcal{C}$, respectively. Such quantum contractions are defined by the usual Inönü–Wigner transformations of generators, providing the Lie algebra contractions $so(4, 2) \to \mathcal{G}$ and $so(4, 2) \to \mathcal{C}$, together with a mapping on the deformation parameter z that ensures the convergence of the classical r-matrix [24, 38]. We stress that computations can be done once $U_z(\mathcal{WP}) \subset U_z(so(4, 2))$ are expressed in the standard conformal basis presented in the previous section.

6.1. Twisted conformal Galilean algebra $U_z(\mathcal{G})$

The quantum *speed–space* contraction that starting from $U_z(so(4, 2))$ leads to $U_z(\mathcal{G})$ is defined by

$$\begin{array}{llll} J_i \to J_i & P_0 \to P_0 & C_0 \to C_0 & D \to D \\ P_i \to \varepsilon P_i & K_i \to \varepsilon K_i & C_i \to \varepsilon C_i & z \to \varepsilon^{-2} z \end{array} \tag{6.1}$$

where ε is the contraction parameter which is related to the speed of light c through $\varepsilon = 1/c$.

Once the map (6.1) is applied to *any* element associated with $U_z(so(4, 2))$, the limit $\varepsilon \to 0$ (or $c \to \infty$) gives rise to that corresponding to $U_z(\mathcal{G})$. Explicitly, the contraction of the classical r-matrix (4.2) originates the r-matrix

$$r = z(K_1 \wedge P_1 + K_2 \wedge P_2 + K_3 \wedge P_3). \tag{6.2}$$

The coproduct and non-vanishing commutation relations of $U_z(\mathcal{G})$ turn out to be

$$\begin{array}{l} \Delta(X) = X \otimes 1 + 1 \otimes X \qquad X \in \{J_i, K_i, P_\mu, C_i\} \\ \Delta(D) = D \otimes 1 + 1 \otimes D + z(K_1 \otimes P_1 + K_2 \otimes P_2 + K_3 \otimes P_3) \\ \Delta(C_0) = C_0 \otimes 1 + 1 \otimes C_0 - z(C_1 \otimes P_1 + C_2 \otimes P_2 + C_3 \otimes P_3) \end{array} \tag{6.3}$$

$$\begin{array}{lll} [J_i, J_j] = \epsilon_{ijk} J_k & [J_i, K_j] = \epsilon_{ijk} K_k & [J_i, P_j] = \epsilon_{ijk} P_k \\ [J_i, C_j] = \epsilon_{ijk} C_k & [K_i, P_0] = P_i & [K_i, C_0] = C_i \\ [P_0, C_0] = D & [P_0, C_i] = -K_i & [P_i, C_0] = K_i \\ [P_i, D] = -P_i & [P_0, D] = -P_0 - \frac{z}{2}\mathbf{P}^2 & \\ [C_i, D] = C_i & [C_0, D] = C_0 - \frac{z}{2}\mathbf{K}^2 & \end{array} \tag{6.4}$$

where $\mathbf{P}^2 = P_1^2 + P_2^2 + P_3^2$ and $\mathbf{K}^2 = K_1^2 + K_2^2 + K_3^2$. Now commutation relations involving conformal transformations are rather simplified.

Under contraction the Φ-factors (5.5) and twisting element $\mathcal{F}$ (5.6) reduce to

$$\begin{array}{l} \Phi_{\mathcal{J}} = \Phi_{\mathcal{A}} = \exp\left(-\frac{z}{2} K_1 \otimes P_1\right) \\ \Phi_{\mathcal{E}} = \exp(-z K_2 \otimes P_2) \exp(-z K_3 \otimes P_3) \\ \mathcal{F} = \exp(-z K_3 \otimes P_3) \exp(-z K_2 \otimes P_2) \exp(-z K_1 \otimes P_1) \end{array} \tag{6.5}$$

which, in turn, determine the universal $\mathcal{R}$-matrix of $U_z(\mathcal{G})$, which is simply the exponential of the classical r-matrix (6.2)

$$\mathcal{R} = \mathrm{e}^r = \exp(z K_1 \wedge P_1) \exp(z K_2 \wedge P_2) \exp(z K_3 \wedge P_3). \tag{6.6}$$

Note that the carrier subalgebra of $U_z(\mathcal{G})$ is a 6D Abelian subalgebra generated by $\{P_i, K_i\}$ and that the Weyl–Galilean subalgebra $\mathcal{WG}$ spanned by $\{J_i, P_\mu, K_i\}$ remains as a Hopf subalgebra, $U_z(\mathcal{WG})$, of $U_z(\mathcal{G})$.

6.2. Twisted conformal Carroll algebra $U_z(\mathcal{C})$

The second non-relativistic limit $U_z(so(4, 2)) \to U_z(\mathcal{C})$ corresponds to the quantum *speed–time* contraction defined by the map

$$\begin{array}{llll} J_i \to J_i & P_i \to P_i & C_i \to C_i & D \to D \\ P_0 \to \varepsilon P_0 & K_i \to \varepsilon K_i & C_0 \to \varepsilon C_0 & z \to \varepsilon^{-1} z. \end{array} \tag{6.7}$$

The classical r-matrix for $U_z(\mathcal{C})$ reads

$$r = z(D \wedge P_0 + K_1 \wedge P_1 + K_2 \wedge P_2 + K_3 \wedge P_3). \tag{6.8}$$

The coproduct and non-vanishing commutation relations of $U_z(\mathcal{C})$ are given by

$$\begin{aligned} &\Delta(X) = X \otimes 1 + 1 \otimes X \qquad X \in \{J_i, P_\mu\} \\ &\Delta(Y) = Y \otimes \mathrm{e}^{zP_0} + 1 \otimes Y \qquad Y \in \{K_i, C_0\} \\ &\Delta(D) = D \otimes \mathrm{e}^{zP_0} + 1 \otimes D + zK_1 \otimes \mathrm{e}^{zP_0} P_1 + zK_2 \otimes \mathrm{e}^{zP_0} P_2 + zK_3 \otimes \mathrm{e}^{zP_0} P_3 \\ &\Delta(C_i) = C_i \otimes \mathrm{e}^{zP_0} + 1 \otimes C_i - zC_0 \otimes \mathrm{e}^{zP_0} P_i + z\epsilon_{ijk} K_j \otimes \mathrm{e}^{zP_0} J_k \end{aligned} \tag{6.9}$$

$$\begin{aligned} &[J_i, J_j] = \epsilon_{ijk} J_k \qquad [J_i, K_j] = \epsilon_{ijk} K_k \qquad [J_i, P_j] = \epsilon_{ijk} P_k \\ &[J_i, C_j] = \epsilon_{ijk} C_k \qquad [K_i, P_i] = \frac{1}{z}(\mathrm{e}^{zP_0} - 1) \qquad [K_i, C_i] = C_0 - \frac{z}{2}\mathbf{K}^2 \\ &[P_0, D] = \frac{1}{z}(1 - \mathrm{e}^{zP_0}) \qquad [P_i, D] = -\mathrm{e}^{zP_0} P_i \qquad [P_0, C_i] = -K_i \\ &[C_0, D] = C_0 - \frac{z}{2}\mathbf{K}^2 \qquad [C_i, D] = C_i + zK_i D \qquad [P_i, C_0] = K_i \\ &[P_i, C_j] = -\delta_{ij} D + \epsilon_{ijk}\, \mathrm{e}^{zP_0} J_k \qquad [C_i, C_j] = z(K_i C_j - K_j C_i). \end{aligned} \tag{6.10}$$

The Weyl–Carroll subalgebra $\mathcal{WC}$, $U_z(\mathcal{WC})$, again remains as a Hopf subalgebra of $U_z(\mathcal{C})$.

On the other hand, although divergences within the Φ-factors (5.5) arise under the map (6.7), both the twisting element (5.6) and the corresponding universal $\mathcal{R}$-matrix have well-defined limits given by

$$\begin{aligned} \mathcal{F} &= \exp(-zK_3 \otimes P_3) \exp(-zK_2 \otimes P_2) \exp(-zK_1 \otimes P_1) \exp(-zD \otimes P_0) \\ \mathcal{R} &= \exp(-zP_3 \otimes K_3) \exp(-zP_2 \otimes K_2) \exp(-zP_1 \otimes K_1) \exp(-zP_0 \otimes D) \\ &\quad \times \exp(zD \otimes P_0) \exp(zK_1 \otimes P_1) \exp(zK_2 \otimes P_2) \exp(zK_3 \otimes P_3). \end{aligned} \tag{6.11}$$

We stress that, for this dimension, the conformal Carroll algebra is isomorphic to a (4+1)D Poincaré algebra $iso(4, 1)$; for a direct relationship a well-adapted basis is the (4+1)D version of the null-plane basis (3.4). In such a basis, $U_z(\mathcal{C})$ turns out to be the (4 + 1)D version of the (3 + 1)D null-plane quantum Poincaré algebra obtained in [11, 12]. We also recall that the twisting element (6.11) for the (3 + 1)D case was deduced in [39] by following a constructive instead of a contraction method.

7. Difference-differential massless Klein–Gordon equation

Non-standard quantum algebras have already been shown to be symmetry algebras of several difference-differential equations such as the (1 + 1)D Schrödinger [40], (1 + 1)D wave and 2D Laplace [22], and (3+1)D wave and 4D Laplace equations [26]. In all these cases, discretization arises in *one* (space or time) direction on a uniform lattice, for which the deformation parameter can be interpreted as the lattice constant. Our aim now is to show that the generators of $U_z(so(4, 2))$ can be expressed as discrete symmetries of a (3 + 1)D difference-differential massless Klein–Gordon (KG) equation which, in turn, provides a physical interpretation for the deformation parameter z that is slightly different with respect to the aforementioned known cases.

Let us consider the Minkowskian spacetime with coordinates $x = (x^0, x^1, x^2, x^3)$ and metric $(g_{\mu\nu}) = \mathrm{diag}(+, -, -, -)$ as introduced in section 3. As is well known a differential realization of its conformal Lie algebra $so(4, 2)$ with Lie brackets (3.1) is given by

$$\begin{aligned} &J_i = \epsilon_{ijk} x^k \partial_j \qquad P_\mu = \partial_\mu \qquad K_i = -x^i \partial_0 - x^0 \partial_i \qquad D = -x^\mu \partial_\mu - \ell \\ &C_0 = \tfrac{1}{2} x^\mu x_\mu \partial_0 - x^0 (x^\mu \partial_\mu + \ell) \qquad C_i = \tfrac{1}{2} x^\mu x_\mu \partial_i + x^i (x^\mu \partial_\mu + \ell) \end{aligned} \tag{7.1}$$

where $\partial_\mu = \partial/\partial x^\mu$, $x^\mu x_\mu = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2$ and ℓ is called the conformal weight. Under this realization, the Casimir operator of the Poincaré subalgebra, $C = P_0^2 - \mathbf{P}^2$, provides the d'Alembertian $\Box$ which leads to the massless KG equation:

$$\Box\phi(x) \equiv \left(\partial_0^2 - \partial_1^2 - \partial_2^2 - \partial_3^2\right)\phi(x) = 0. \tag{7.2}$$

The following commutation relations between the d'Alembertian and the operators (7.1)

$$\begin{array}{ll} [P_\mu, \Box] = [J_i, \Box] = [K_i, \Box] = 0 & [D, \Box] = 2\Box \\ [C_0, \Box] = 2x^0\Box + 2(\ell - 1)\partial_0 & [C_i, \Box] = -2x^i\Box + 2(\ell - 1)\partial_i \end{array} \tag{7.3}$$

show that $so(4,2)$ is a Lie symmetry algebra of the massless KG equation whenever $\ell = 1$. That is, its operators carry solutions of this equation into solutions.

Lie symmetries of the KG equation are deeply related with the commutation relations $[\partial_\mu, x^\nu] = \delta_{\mu\nu}$ (or $[\partial_\mu, x_\nu] = g_{\mu\nu}$). Hence, a procedure to obtain a difference-differential version of (7.2) with $U_z(so(4,2))$-symmetry is first to deduce some momentum $\hat{\mathrm{p}}_\mu = \hat{\mathrm{p}}_\mu(\partial, x)$ and position $\hat{\mathrm{x}}^\mu = \hat{\mathrm{x}}^\mu(\partial, x)$ operators fulfilling

$$[\hat{\mathrm{p}}_\mu, \hat{\mathrm{x}}^\nu] = \delta_{\mu\nu} \qquad [\hat{\mathrm{p}}_\mu, \hat{\mathrm{x}}_\nu] = g_{\mu\nu} \qquad [\hat{\mathrm{x}}^\mu, \hat{\mathrm{x}}^\nu] = 0 \qquad [\hat{\mathrm{p}}_\mu, \hat{\mathrm{p}}_\nu] = 0 \tag{7.4}$$

which in the limit $z \to 0$ should give $\hat{\mathrm{p}}^\mu \to \partial_\mu$, $\hat{\mathrm{x}}^\mu \to x^\mu$, and second to express the generators of $U_z(so(4,2))$ in terms of such operators.

The starting point in this construction comes from the nonlinear change of basis (5.1) applied in section 5 which naturally includes *discrete derivatives*. Explicitly, if we write the generators $T_\pm$, T_a (3.6) as the operators $\hat{\mathrm{p}}^\mu$:

$$T_+ = \tfrac{1}{2}(\hat{\mathrm{p}}_0 + \hat{\mathrm{p}}_1) \qquad T_- = \tfrac{1}{2}(\hat{\mathrm{p}}_0 - \hat{\mathrm{p}}_1) \qquad T_2 = \tfrac{1}{2}(\hat{\mathrm{p}}_2 - \hat{\mathrm{p}}_3) \qquad T_3 = \tfrac{1}{2}(\hat{\mathrm{p}}_2 + \hat{\mathrm{p}}_3) \tag{7.5}$$

and consider the translation generators P_μ introduced by means of the nonlinear map (5.1) as the derivatives ∂_μ, then the transformed expressions for $T_\pm$, T_a given in (5.2) allow us to make the following ansatz for the $\hat{\mathrm{p}}_\mu$ operators:

$$\begin{aligned} \hat{\mathrm{p}}_0 + \hat{\mathrm{p}}_1 &= \frac{1}{z}(1 - \mathrm{e}^{-z\partial_0}\,\mathrm{e}^{-z\partial_1}) - z\,\mathrm{e}^{-z\partial_0}\,\mathrm{e}^{z\partial_1}\left(\partial_2^2 + \partial_3^2\right) \\ \hat{\mathrm{p}}_0 - \hat{\mathrm{p}}_1 &= \frac{1}{z}(1 - \mathrm{e}^{-z\partial_0}\,\mathrm{e}^{z\partial_1}) \\ \hat{\mathrm{p}}_2 \pm \hat{\mathrm{p}}_3 &= \mathrm{e}^{-z\partial_0}\,\mathrm{e}^{z\partial_1}(\partial_2 \pm \partial_3). \end{aligned} \tag{7.6}$$

In other words, $\sigma_\pm = -z(\partial_0 \pm \partial_1)$ and the elements T_- and T are directly two discrete derivatives along two basic directions $x^0 - x^1$ and $x^0 + x^1$, respectively.

Next, by requiring commutators (7.4) to be satisfied, we find the corresponding $\hat{\mathrm{x}}^\mu$ operators:

$$\begin{aligned} &\hat{\mathrm{x}}^0 + \hat{\mathrm{x}}^1 = (x^0 + x^1)\,\mathrm{e}^{z\partial_0}\,\mathrm{e}^{z\partial_1} \\ &\hat{\mathrm{x}}^0 - \hat{\mathrm{x}}^1 = \{x^0 - x^1 + 2z(x^2\partial_2 + x^3\partial_3)\}\,\mathrm{e}^{z\partial_0}\,\mathrm{e}^{-z\partial_1} + z^2(x^0 + x^1)\,\mathrm{e}^{z\partial_0}\,\mathrm{e}^{z\partial_1}\left(\partial_2^2 + \partial_3^2\right) \\ &\hat{\mathrm{x}}^2 \pm \hat{\mathrm{x}}^3 = (x^2 \pm x^3)\,\mathrm{e}^{z\partial_0}\,\mathrm{e}^{-z\partial_1} + z(x^0 + x^1)\,\mathrm{e}^{z\partial_0}\,\mathrm{e}^{z\partial_1}(\partial_2 \pm \partial_3). \end{aligned} \tag{7.7}$$

Therefore, from (7.6) and (7.7) we find that

$$\begin{aligned} \hat{\mathrm{p}}_0 &= \frac{1}{z}(1 - \mathrm{e}^{-z\partial_0}\cosh z\partial_1) - \frac{z}{2}\,\mathrm{e}^{-z\partial_0}\,\mathrm{e}^{z\partial_1}\left(\partial_2^2 + \partial_3^2\right) \\ \hat{\mathrm{p}}_1 &= \frac{1}{z}\,\mathrm{e}^{-z\partial_0}\sinh z\partial_1 - \frac{z}{2}\,\mathrm{e}^{-z\partial_0}\,\mathrm{e}^{z\partial_1}\left(\partial_2^2 + \partial_3^2\right) \\ \hat{\mathrm{p}}_a &= \mathrm{e}^{-z\partial_0}\,\mathrm{e}^{z\partial_1}\,\partial_a \end{aligned}$$

$$
\begin{aligned}
\hat{x}^0 &= x^0\,e^{z\partial_0}\cosh z\partial_1 + x^1\,e^{z\partial_0}\sinh z\partial_1 + z(x^2\partial_2 + x^3\partial_3)\,e^{z\partial_0}\,e^{-z\partial_1} \\
&\qquad + \frac{z^2}{2}(x^0 + x^1)\,e^{z\partial_0}\,e^{z\partial_1}\left(\partial_2^2 + \partial_3^2\right) \\
\hat{x}^1 &= x^1\,e^{z\partial_0}\cosh z\partial_1 + x^0\,e^{z\partial_0}\sinh z\partial_1 - z(x^2\partial_2 + x^3\partial_3)\,e^{z\partial_0}\,e^{-z\partial_1} \\
&\qquad - \frac{z^2}{2}(x^0 + x^1)\,e^{z\partial_0}e^{z\partial_1}\left(\partial_2^2 + \partial_3^2\right) \\
\hat{x}^a &= x^a\,e^{z\partial_0}\,e^{-z\partial_1} + z(x^0 + x^1)\,e^{z\partial_0}\,e^{z\partial_1}\,\partial_a
\end{aligned}
\tag{7.8}
$$

where $a = 2, 3$. If we now apply the maps $\partial_\mu \mapsto \hat{p}_\mu, x^\mu \mapsto \hat{x}^\mu$ to the initial differential operators (7.1) and KG equation (7.2), we obtain difference-differential symmetries $X(\partial_\mu, x^\mu) \mapsto \hat{X}(\hat{p}_\mu, \hat{x}^\mu)$ of the following difference-differential analogue of the massless KG equation:

$$
\Box_z\phi(x) = \left\{\left(\frac{1 - e^{-z\partial_0}\,e^{-z\partial_1}}{z}\right)\left(\frac{1 - e^{-z\partial_0}\,e^{z\partial_1}}{z}\right) - e^{-z\partial_0}\,e^{z\partial_1}\left(\partial_2^2 + \partial_3^2\right)\right\}\phi(x) = 0 \tag{7.9}
$$

which is the generalization of the $(2+1)$D case studied in [28]. The transformed commutators (7.3) with $\ell = 1$ show that $U_z(so(4, 2))$ is a symmetry-twisted algebra of this equation. The coproduct for the resulting operators can be deduced from the expressions given in section 4 through the change of basis (3.6)–(3.8). This can be further used in the construction of the composition of several difference-differential massless KG equations.

On the other hand, relations (7.6) and (7.7) indicate that an alternative and natural basis to rewrite the above results is the Minkowskian null-plane one (3.2). Then by denoting

$$
\partial_\pm = \partial_0 \pm \partial_1 \qquad [\partial_\pm, x^\pm] = 1 \qquad [\partial_\pm, x^\mp] = 0 \qquad \hat{p}_\pm = \hat{p}_0 \pm \hat{p}_1 \qquad \hat{x}^\pm = \tfrac{1}{2}(\hat{x}^0 \pm \hat{x}^1) \tag{7.10}
$$

we find the following null-plane momentum and position operators

$$
\begin{aligned}
\hat{p}_+ &= \frac{1}{z}(1 - e^{-z\partial_+}) - z\,e^{-z\partial_-}\left(\partial_2^2 + \partial_3^2\right) & \hat{x}^+ &= x^+\,e^{z\partial_+} \\
\hat{p}_- &= \frac{1}{z}(1 - e^{-z\partial_-}) & \hat{x}^- &= x^-\,e^{z\partial_-} + z(x^2\partial_2 + x^3\partial_3)\,e^{z\partial_-} + z^2x^+\,e^{z\partial_+}\left(\partial_2^2 + \partial_3^2\right) \\
\hat{p}_a &= e^{-z\partial_-}\,\partial_a & \hat{x}^a &= x^a\,e^{z\partial_-} + 2zx^+\,e^{z\partial_+}\,\partial_a
\end{aligned}
\tag{7.11}
$$

that verify (7.4) for $\mu, \nu = +, -, 2, 3$. In this null-plane context the massless KG equation (7.9) is rewritten as

$$
\Box_z\phi(x) = \left\{\left(\frac{1 - e^{-z\partial_+}}{z}\right)\left(\frac{1 - e^{-z\partial_-}}{z}\right) - e^{-z\partial_-}\left(\partial_2^2 + \partial_3^2\right)\right\}\phi(x^+, x^-, x^2, x^3) = 0 \tag{7.12}
$$

where the action of a discrete derivative reads

$$
\frac{1}{z}(1 - e^{-z\partial_+})\phi(x^+, x^-, x^2, x^3) = \frac{1}{z}\{\phi(x^+, x^-, x^2, x^3) - \phi(x^+ - z, x^-, x^2, x^3)\} \tag{7.13}
$$

and similarly for that with ∂_-. Consequently, we clearly obtain an intrinsic discretization on a uniform lattice along the *two* null-plane directions $x^\pm$ where z is the *same* lattice step for both of them; the transverse coordinates x^a remain continuous. Furthermore, the null-plane evolution framework with $x^- = \tau$ (3.2) also provides an interpretation of z as a fundamental constant directly associated with a discretization of the 'time' evolution parameter τ so that the transformed operator $\hat{P}_-(\hat{p}_\mu, \hat{x}^\mu)$ would play the role of a difference-differential Hamiltonian [11].

8. Conclusions and remarks

In this paper we have constructed a new quantum twisted deformation of the conformal algebra of the (3 + 1)D Minkowskian spacetime that generalizes the previously known lower dimensional cases [10, 24]. The twisting element has been established within a conformal null-plane basis. Moreover, expressions in the standard conformal basis have been deduced through a nonlinear map and, among them, we remark the 'compact' form of the universal $\mathcal{R}$-matrix of $U_z(so(4,2))$.

These results have allowed us to perform a study of contractions that relate in a consistent way $U_z(so(4,2))$ with its non-relativistic limits together with their Weyl Hopf subalgebras, classical r-matrices, twisting elements and universal $\mathcal{R}$-matrices. In particular, the twisted conformal Carroll algebra is the (4+1)D version of the so-called null-plane quantum Poincaré algebra [11, 12].

As an application, it has been also shown that $U_z(so(4,2))$ is the symmetry Hopf algebra of a difference-differential massless KG equation, for which the symmetry operators have been deduced; this has provided an interpretation of the deformation parameter z as the lattice constant in a discretization of the (3 + 1)D Minkowskian spacetime along *two* null-plane directions.

All of these results are summarized in the following diagram, where vertical arrows indicate contractions.

Twisted conformal Galilean algebra	$U_z(\mathcal{WG}) \subset U_z(\mathcal{G})$
	$\uparrow$
Twisted conformal Minkowskian algebra	$U_z(\mathcal{WP}) \subset U_z(so(4,2)) \leftrightarrow \Box_z\phi(x) = 0$
	$\downarrow$
Twisted conformal Carroll/Null-plane Poincaré	$U_z(\mathcal{WC}) \subset U_z(\mathcal{C}) \simeq U_z(iso(4,1))$

To end with we would like to comment on a twisting deformation of $sl(4)$ presented by Kulish and Lyakhovsky (KL) in [30] as well as on the classical r-matrices for $so(4,2)$ obtained by Lukierski, Lyakhovsky and Mozrzymas (LLM) in [25]

The KL deformation is also based on an 8D carrier subalgebra $\mathbf{L}$ of $sl(4)$, which, as a *complex* algebra, is isomorphic to $so(6)$ (although we have dealt with the real form $so(4,2)$). When both carrier algebras, $\mathbf{L}$ and $\mathcal{L}$ (3.11), are considered over the complex numbers, there is an isomorphism $\mathbf{L} \simeq \mathcal{L}$ defined by the following change of basis:

$$\begin{aligned} &H_{23} = H \equiv H_+ && H_{14} = H_{\lambda_0^\perp} \equiv H_- \\ &E_{23} = L_{\lambda_0} \equiv -\frac{1}{2z}T_+ && E_{14} = L_{\lambda_0^\perp} \equiv -2zT_- \\ &E_{13} = L_{\lambda_0^\perp - \tilde{\lambda}'} \equiv \frac{1}{\sqrt{2}}(T_3 - \mathrm{i}T_2) && E_{24} = L_{\lambda_0^\perp - \lambda'} \equiv \frac{1}{\sqrt{2}}(T_3 + \mathrm{i}T_2) \\ &E_{12} = L_{\lambda'} \equiv -z\sqrt{2}(B_3 - \mathrm{i}B_2) && E_{34} = -L_{\tilde{\lambda}'} \equiv z\sqrt{2}(B_3 + \mathrm{i}B_2) \end{aligned} \tag{8.1}$$

where the KL generators are written on the lhs in two notations; starting from the commutation relations of $\mathbf{L}$ we recover those of $\mathcal{L}$ (3.11). If now we apply this change of basis to the KL twisting element, here written as

$$\mathcal{F}_{\mathrm{KL}} = \Psi_{\mathcal{BJ}}\Psi_{\mathcal{E}'}\Psi_{\mathcal{E}}\Psi_{\mathcal{J}_\perp} \tag{8.2}$$

we also find that this coincides exactly with $\mathcal{F}$ (4.4), for which the Φ-factors (4.5) are related to the KL Ψ-factors by means of

$$\Psi_{\mathcal{J}_\perp} \equiv \Phi_{\mathcal{J}} \qquad \Psi_{\mathcal{E}'}\Psi_{\mathcal{E}} \equiv \Phi_{\mathcal{E}} \qquad \Psi_{\mathcal{BJ}} \equiv \Phi_{\mathcal{A}} \tag{8.3}$$

provided that

$$\sigma_{14} \equiv \sigma_- \qquad \sigma_{\mathcal{B}} \equiv \sigma_+ \qquad E_{\mathcal{B}} \equiv 2zT. \tag{8.4}$$

Consequently, the twisting **L** and $\mathcal{L}$ algebras are identified.

The LLM deformations are provided by classical r-matrices also defined on carrier subalgebras of $sl(4) \simeq so(6)$ which give rise to κ-deformations of $\mathcal{P}, \mathcal{WP}$ and $so(4,2)$ in a conformal basis by imposing reality conditions. The so-called light-like κ-Poincaré r-matrix is expressed in our basis (3.1) as

$$r_{LLM} = zK_3 \wedge P_0 + z(K_1 \wedge P_1 + K_2 \wedge P_2 + K_3 \wedge P_3) + z(J_1 \wedge P_2 - J_2 \wedge P_1) \tag{8.5}$$

where $z = 1/\kappa$. By comparing r_{LLM} with our r (4.2), we find that the former is associated with the boost generator K_3 instead of K_1, but the relevant difference among them lies in the replacement $K_3 \wedge P_0 \leftrightarrow D \wedge P_0$. In fact, r (4.2) may be seen as a light-like κ-Weyl–Poincaré r-matrix. Another more involved LLM classical r-matrix is the κ-Weyl matrix that reads

$$\begin{aligned}\hat{r}_{LLM} &= r_{LLM} + \frac{z}{2}(D - K_3) \wedge (P_0 + P_3)\\ &= \frac{z}{2}K_3 \wedge P_0 + \frac{z}{2}D \wedge (P_0 + P_3) + z\left(K_1 \wedge P_1 + K_2 \wedge P_2 + \frac{1}{2}K_3 \wedge P_3\right)\\ &\quad + z(J_1 \wedge P_2 - J_2 \wedge P_1).\end{aligned} \tag{8.6}$$

Nevertheless, the r-matrix (4.2) can be identified within the multiparameter LLM r-matrix $r_+^{(1)}(\alpha, \beta, \gamma, \delta)$ by setting $\alpha = \beta = \gamma = 0$ and $\delta = 1/2$.

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